A note over para complex torsion-free affine connection on tangent bundle

Mehmet Tekkoyun *
Department of Mathematics, Pamukkale University,
20070 Denizli, Turkey
Ali Görgülü †
Department of Mathematics, Eskişehir Osmangazi University,
26480 Eskişehir, Turkey

February 24, 2009

Abstract

The goal of this paper is to introduce the lifting theory that has an important role in geometry. Therefore, using the lifts of differential geometric structures we show that tangent bundle TM of paracomplex manifold M admits para-complex torsion-free affine connection.

Keywords: paracomplex structure, paracomplex manifold, lift theory, para-complex torsion-free affine connection.

1 Introduction

In differential geometry, lifting method has an important tool. So, using lift function it may be possible to generalize to differentiable structures on any manifold or space to its extensions. There are many books and studies about lift theory. Some of them are given in [1-7]. Lifts of differential geometric elements defined on any manifold M to tangent manifold TM

*tekkoyun@pau.edu.tr
†agorgulu@ogu.edu.tr

has been obtained by Yano and Ishihara [7]. Para-Complex geometry are introduced by Schäfer [8] and, Cruceanu and others [9]. Complex and paracomplex lift analogues of the geometric structures had been introduced by Tekkoyun [3,4,5] and Civelek [3,4]. Also, complex and paracomplex lift analogues of the Lagrangian and Hamiltonian systems in classical mechanics were made by Tekkoyun and Görgülü [6]. In this study, firstly, it is recall vertical and complete lifts of fundamental structures in geometry. Then, we deduce that TM, tangent bundle of paracomplex manifold M, admits para-complex torsion-free affine connection.

Throughout this paper, all maps will be understood to be differentiable of class C^∞ and the sum is taken over repeated indices. Also, the indices α, β are assumed $1 \leq \alpha, \beta \leq m$.

1.1 Paracomplex Geometry

A tensor field J of type (1,1) on M such that $J^2 = I$ is called an *almost product structure* J on 2m-dimensional manifold M. Then, *almost product manifold* is said to be the pair (M, J). An *almost paracomplex manifold* is an almost product manifold (M, J) such that the two eigenbundles $T^{\pm}M$ associated to the eigenvalues ± 1 of J, respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even. Equivalently, a splitting of the tangent bundle TM of manifold M, into the Whitney sum of two subbundles on $T^{\pm}M$ of the same fiber dimension is called an *almost paracomplex structure* on M. An almost paracomplex structure on manifold M may alternatively be defined as a G- structure on M with structural group $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$.

If the G- structure defined by the tensor field J is integrable, we call that an almost paracomplex manifold (M, J) is a *paracomplex manifold.*

Let (x^α, y^α) be a real coordinate system on a neighborhood U of any point p of M, and $\{(\frac{\partial}{\partial x^\alpha})_p, (\frac{\partial}{\partial y^\alpha})_p\}$ and $\{(dx^\alpha)_p, (dy^\alpha)_p\}$ natural bases over $\mathbf{R}$ of the tangent space and the cotangent space T_pM and T_p^*M of M, respectively. Then we explain as

$$J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha},\ J(\frac{\partial}{\partial y^\alpha}) = \frac{\partial}{\partial x^\alpha}$$

and

$$J^*(dx^\alpha) = -dy^\alpha,\ J^*(dy^\alpha) = -dx^\alpha.$$

Let $z^{\alpha}=x^{\alpha}+\mathbf{j}y^{\alpha}$, $\overline{z}^{\alpha}=x^{\alpha}-\mathbf{j}y^{\alpha}$, $\mathbf{j}^2=1$, be a paracomplex local coordinate system on a neighborhood U of any point p of M. We express the vector fields and dual covector fields as:

$$(\frac{\partial}{\partial z^{\alpha}})_p=\frac{1}{2}\{(\frac{\partial}{\partial x^{\alpha}})_p-\mathbf{j}(\frac{\partial}{\partial y^{\alpha}})_p\},\ (\frac{\partial}{\partial \overline{z}^{\alpha}})_p=\frac{1}{2}\{(\frac{\partial}{\partial x^{\alpha}})_p+\mathbf{j}(\frac{\partial}{\partial y^{\alpha}})_p\},$$

$$(dz^{\alpha})_p=(dx^{\alpha})_p+\mathbf{j}(dy^{\alpha})_p,\ (d\overline{z}^{\alpha})_p=(dx^{\alpha})_p-\mathbf{j}(dy^{\alpha})_p.$$

which symbolize the bases of the tangent space and cotangent space T_pM and T_p^*M of M, respectively. Then, using $\mathbf{j}^2=1$ it is found

$$J(\frac{\partial}{\partial z^{\alpha}})=-\mathbf{j}\frac{\partial}{\partial z^{\alpha}},\ J(\frac{\partial}{\partial \overline{z}^{\alpha}})=\mathbf{j}\frac{\partial}{\partial \overline{z}^{\alpha}}.$$

The dual map J^* of the cotangent space T_p^*M of manifold M at any point p satisfies $J^{*2}=I$. Thus, using $\mathbf{j}^2=1$, it is computed by

$$J^*(dz^{\alpha})=-\mathbf{j}dz^{\alpha},\ J^*(d\overline{z}^{\alpha})=\mathbf{j}d\overline{z}^{\alpha}.$$

If vector space T_pM is the set of tangent vectors $Z_p=Z^{\alpha}(\frac{\partial}{\partial z^{\alpha}})_p+\overline{Z}^{\alpha}(\frac{\partial}{\partial \overline{z}^{\alpha}})_p$,for each $p\in M$, then TM is the union of T_pM. Thus, tangent bundle of a paracomplex manifold M is (TM,τ_M,M), where canonical projection τ_M is $\tau_M:TM\to M$ ($\tau_M(Z_p)=p$) and, also this map are surjective submersion. After now, coordinates $\left\{z^{\alpha},\overline{z}^{\alpha},\acute{z^{\alpha}},\acute{\overline{z}^{\alpha}}\right\}$ are local coordinates for TM.

2 Lifting theory of paracomplex structures

2.1 Lifts of function

The function $f^v\in\mathcal{F}(TM)$ given by

$$f^v=f\circ\tau_M$$

is called *vertical lift* of paracomplex function $f\in\mathcal{F}(M)$ to TM, where $\tau_M:TM\to M$ canonical projection. We get $rang(f^v)=rang(f)$, since

$$f^v(Z_p)=f(\tau_M(Z_p))=f(p),\quad \forall\, Z_p\in TM.$$

The *complete lift* of paracomplex function $f\in\mathcal{F}(M)$ to TM is the function $f^c\in\mathcal{F}(TM)$ given by

$$f^c=\acute{z^{\alpha}}(\frac{\partial f}{\partial z^{\alpha}})^v+\acute{\overline{z}^{\alpha}}(\frac{\partial f}{\partial \overline{z}^{\alpha}})^v,$$

where $(z^{\alpha}, \overline{z}^{\alpha}, \acute{z}^{\alpha}, \acute{\overline{z}}^{\alpha})$ are the local coordinates of a chart-domain $TU \subset TM$. Further, for $Z_p \in TM$ it holds

$$f^c(Z_p) = \acute{z}^{\alpha}(Z_p)(\frac{\partial f}{\partial z^{\alpha}})^v(p) + \acute{\overline{z}}^{\alpha}(Z_p)(\frac{\partial f}{\partial \overline{z}^{\alpha}})^v(p).$$

The general properties of the vertical and complete lifts of paracomplex functions are as follows:

$$\begin{array}{ll} i) & (f.g)^v = f^v.g^v, (f+g)^v = f^v + g^v, \\ ii) & (f.g)^c = f^c.g^v + f^v.g^c, (f+g)^c = f^c + g^c, \end{array}$$

for all $f, g \in \mathcal{F}(M)$.

2.2 Lifts of vector field

In this subsection, we assume that vector field Z is $Z = Z^{\alpha}\frac{\partial}{\partial z^{\alpha}} + \overline{Z}^{\alpha}\frac{\partial}{\partial \overline{z}^{\alpha}}$. The vector field $Z^v \in \chi(TM)$ determined by

$$Z^v(f^c) = (Zf)^v, \quad \forall f \in \mathcal{F}(M)$$

is the *vertical lift* of a vector field $Z \in \chi(M)$ to TM. Then we get

$$Z^v = (Z^{\alpha})^v \frac{\partial}{\partial \acute{z}^{\alpha}} + (\overline{Z}^{\alpha})^v \frac{\partial}{\partial \acute{\overline{z}}^{\alpha}}.$$

The *complete lift* of a vector field $Z \in \chi(M)$ to TM is the vector field $Z^c \in \chi(TM)$ given by

$$Z^c(f^c) = (Zf)^c, \quad \forall f \in \mathcal{F}(M).$$

Clearly, it is gotten

$$Z^c = (Z^{\alpha})^v \frac{\partial}{\partial z^{\alpha}} + (\overline{Z}^{\alpha})^v \frac{\partial}{\partial \overline{z}^{\alpha}} + (Z^{\alpha})^c \frac{\partial}{\partial \acute{z}^{\alpha}} + (\overline{Z}^{\alpha})^c \frac{\partial}{\partial \acute{\overline{z}}^{\alpha}}.$$

The vertical and complete lifts of paracomplex vector fields have the following general properties:

$$\begin{array}{ll} i) & (X+Y)^v = X^v + Y^v, (X+Y)^c = X^c + Y^c, \\ ii) & (fX)^v = f^v X^v, (fX)^c = f^c X^v + f^v X^c, \\ iii) & X^v(f^v) = 0, X^c(f^v) = X^v(f^c) = (Xf)^v, X^c(f^c) = (Xf)^c, \\ iv) & [X^v, Y^v] = 0, [X^v, Y^c] = [X^c, Y^v] = [X,Y]^v, [X^c, Y^c] = [X,Y]^c \\ v) & (\frac{\partial}{\partial z^{\alpha}})^c = \frac{\partial}{\partial z^{\alpha}}, (\frac{\partial}{\partial \overline{z}^{\alpha}})^c = \frac{\partial}{\partial \overline{z}^{\alpha}}, (\frac{\partial}{\partial z^{\alpha}})^v = \frac{\partial}{\partial \acute{z}^{\alpha}}, (\frac{\partial}{\partial \overline{z}^{\alpha}})^v = \frac{\partial}{\partial \acute{\overline{z}}^{\alpha}}, \end{array}$$

for all $f \in \mathcal{F}(M)$, $X, Y, Z \in \chi(M)$, $\chi(U) = Sp\left\{\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \overline{z}^{\alpha}}\right\}$, $\chi(TU) = Sp\left\{\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \overline{z}^{\alpha}}, \frac{\partial}{\partial \acute{z}^{\alpha}}, \frac{\partial}{\partial \acute{\overline{z}}^{\alpha}}\right\}$.

2.3 Lifts of 1-form

In this subsection, we consider that the paracomplex 1-form ω is $\omega = \omega_\alpha dz^\alpha + \overline{\omega}_\alpha d\overline{z}^\alpha$. The 1-form $\omega^v \in \chi^*(TM)$ explained by

$$\omega^v(Z^c) = (\omega Z)^v, \quad \forall Z \in \chi(M),$$

is said to be the *vertical lift* of a 1-form $\omega \in \chi^*(M)$ to TM.

Then, we have

$$\omega^v = (\omega_\alpha)^v dz^\alpha + (\overline{\omega}_\alpha)^v d\overline{z}^\alpha.$$

The *complete lift* of a 1-form $\omega \in \chi^*(M)$ to TM is the 1-form $\omega^c \in \chi^*(TM)$ given by

$$\omega^c(Z^c) = (\omega Z)^c, \quad \forall Z \in \chi(M).$$

Hence, we compute

$$\omega^c = (\omega_\alpha)^c dz^\alpha + (\overline{\omega}_\alpha)^c d\overline{z}^\alpha + (\omega_\alpha)^v d\acute{z}^\alpha + (\overline{\omega}_\alpha)^v d\acute{\overline{z}}^\alpha.$$

The properties of the vertical and complete lifts of paracomplex 1-forms are as follows:

$$\begin{array}{ll} i) & (f\omega)^v = f^v\omega^v, (f\omega)^c = f^c\omega^v + f^v\omega^c, \\ ii) & (\omega+\theta)^v = \omega^v + \theta^v, (\omega+\theta)^c = \omega^c + \theta^c \\ iii) & \omega^v(Z^v) = 0, \omega^c(Z^v) = \omega^v(Z^c) = (\omega Z)^v, \omega^c(Z^c) = (\omega Z)^c, \\ iv) & (dz^\alpha)^c = \overline{d}\acute{z}^\alpha, (d\overline{z}^\alpha)^c = \overline{d}\acute{\overline{z}}^\alpha, (dz^\alpha)^v = \overline{d}z^\alpha, (d\overline{z}^\alpha)^v = \overline{d}\overline{z}^\alpha, \end{array}$$

for all $f \in \mathcal{F}(M)$, $Z \in \chi(M)$, $\omega, \theta \in \chi^*(M)$, $\chi^*(U) = Sp\{dz^\alpha, d\overline{z}^\alpha\}$, $\chi^*(TU) = Sp\left\{dz^\alpha, d\overline{z}^\alpha, d\acute{z}^\alpha, d\acute{\overline{z}}^\alpha\right\}$, $\overline{d}$ denotes the differential operator on TM.

2.4 Lifts of tensor fields of type (1,1)

The *complete lift* of a paracomplex tensor field of type (1,1) $F \in \Im_1^1(M)$ to TM is the tensor field $F^c \in \Im_1^1(TM)$ given by

$$F^c(Z^c) = (FZ)^c, \quad \forall Z \in \chi(M).$$

The complete lift of the paracomplex tensor field of type (1,1) F is

$$\begin{aligned} F^c = {} & (F_\alpha^\beta)^v \tfrac{\partial}{\partial z^\beta} \otimes dz^\alpha + (F_\alpha^\beta)^c \tfrac{\partial}{\partial \acute{z}^\beta} \otimes dz^\alpha + (F_\alpha^\beta)^v \tfrac{\partial}{\partial \acute{z}^\beta} \otimes d\acute{z}^\alpha \\ & + (\overline{F}_\alpha^\beta)^v \tfrac{\partial}{\partial \overline{z}^\beta} \otimes d\overline{z}^\alpha + (\overline{F}_\alpha^\beta)^c \tfrac{\partial}{\partial \acute{\overline{z}}^\beta} \otimes d\overline{z}^\alpha + (\overline{F}_\alpha^\beta)^v \tfrac{\partial}{\partial \acute{\overline{z}}^\beta} \otimes d\acute{\overline{z}}^\alpha. \end{aligned}$$

2.5 Lift of para-complex structure

The *complete lift* of J being a paracomplex tensor field of type (1,1) is

$$J^c = \mathbf{j}\frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + \mathbf{j}\frac{\partial}{\partial \acute{z}^\alpha} \otimes d\acute{z}^\alpha - \mathbf{j}\frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha - \mathbf{j}\frac{\partial}{\partial \overline{\acute{z}}^\alpha} \otimes d\overline{\acute{z}}^\alpha.$$

Because of $(J^c)^2 = I$, J^c is an almost paracomplex structure for tangent bundle TM.

3 Para-complex torsion-free affine connection on tangent bundle

In this section, we assume that M is an almost paracomplex manifold and TM its tangent bundle. Let Z, W be vector fields and ∇ paracomplex connection, $[\,,]$ Lie bracket on M.

The torsion tensor T on M is defined as

$$T(Z, W) = \nabla_Z W - \nabla_W Z - [Z, W].$$

The torsion-free tensor T^c on TM determined as

$$T^c(Z^c, W^c) = \nabla^c_{Z^c} W^c - \nabla^c_{W^c} Z^c - [Z^c, W^c] = 0$$

is called the *complete lift* of T on M.

The Nijenhuis tensor N_J endowed with paracomplex structure J on M is defined as

$$N_J(Z, W) = [Z, W] - J\,[JZ, W] - J\,[Z, JW] + [JZ, JW]$$

Using almost paracomplex structure J^c on TM, the *complete lift* of N_J is the Nijenhuis tensor $N^c_{J^c}$of J^c and given by

$$N^c_{J^c}(Z^c, W^c) = [Z^c, W^c] - J^c\,[J^c Z^c, W^c] - J^c\,[Z^c, J^c W^c] + [J^c Z^c, J^c W^c]\,.$$

Theorem 1: Let M be almost para-complex manifold and TM its tangent bundle. Every TM fixed with para-complex structure J^c admits an almost para-complex affine connection with torsion

$$N^c_{J^c} = -4T^c$$

where $N^c_{J^c}$ is the Nijenhuis-tensor of almost para- complex structure J^c and T^c is complete lift of the torsion tensor T.

Proof: Let ∇^c be torsion-free connection on TM. We explain $Q^c \in \Gamma((T(T^*M))^2 \otimes T(TM))$ as:

$$4Q^c(X^c, Y^c) := [(\nabla^c_{J^cY^c} J^c)X^c + J^c((\nabla^c_{Y^c} J^c)X^c) + 2J^c((\nabla^c_{X^c} J^c)Y^c)]$$

and furthermore

$$\widetilde{\nabla}^c_{X^c} Y^c = \nabla^c_{X^c} Y^c + Q^c(X^c, Y^c).$$

In this case, we calculate

$$\begin{aligned}
(\widetilde{\nabla}^c_{X^c} J^c)Y^c &= \widetilde{\nabla}^c_{X^c} J^c Y^c - J^c \widetilde{\nabla}^c_{X^c} Y^c \\
&= \nabla^c_{X^c} J^c Y^c + Q^c(X^c, J^c Y^c) - J^c \nabla^c_{X^c} Y^c - J^c Q^c(X^c, Y^c) \\
&= (\nabla^c_{X^c} J^c) Y^c + Q^c(X^c, J^c Y^c) - J^c Q^c(X^c, Y^c)
\end{aligned}$$

Thinking $A(X^c, Y^c) = Q^c(X^c, J^c Y^c) - J^c Q^c(X^c, Y^c)$, we get

$$(\widetilde{\nabla}^c_{X^c} J^c)Y^c = (\nabla^c_{X^c} J^c)Y^c + A(X^c, Y^c).$$

After this, we have to show $A(X^c, Y^c) = -(\nabla^c_{X^c} J^c)Y^c$. Also, we write

$$\begin{aligned}
4Q^c(X^c, J^cY^c) \ &: \ = [(\nabla^c_{Y^c} J^c)X^c + J^c((\nabla^c_{J^cY^c} J^c)X^c) + 2J^c((\nabla^c_{X^c} J^c)J^cY^c)] \\
4J^cQ^c(X^c, Y^c) \ &: \ = [(J^c\nabla^c_{J^cY^c} J^c)X^c + ((\nabla^c_{Y^c} J^c)X^c) + 2((\nabla^c_{X^c} J^c)Y^c)].
\end{aligned}$$

Taking care of $(J^c)^2 = I$ and using $J^c((\nabla^c_{X^c} J^c)J^cY^c) = -(\nabla^c_{X^c} J^c)Y^c$ and at last we compute

$$4A(X^c, Y^c) = 4Q^c(X^c, J^cY^c) - 4J^cQ^c(X^c, Y^c) = -4(\nabla^c_{X^c} J^c)Y^c.$$

Now, we find the torsion of $\widetilde{\nabla}^c$ given as follow:

$$T^{c^{\widetilde{\nabla}^c}}(X^c, Y^c) = T^{c^{\nabla^c}}(X^c, Y^c) + Q^c(X^c, Y^c) - Q^c(Y^c, X^c) = Q^c(X^c, Y^c) - Q^c(Y^c, X^c).$$

Considering the definition of Q^c we have

$$
\begin{aligned}
4T^{c\widetilde{\nabla}^c}(X^c,Y^c) &= 4Q^c(X^c,Y^c)-4Q^c(Y^c,X^c)\\
&= [(\nabla^c_{J^cY^c}J^c)X^c+J^c((\nabla^c_{Y^c}J^c)X^c)+2J^c((\nabla^c_{X^c}J^c)Y^c)]\\
&\quad -[(\nabla^c_{J^cX^c}J^c)Y^c+J^c((\nabla^c_{X^c}J^c)Y^c)+2J^c((\nabla^c_{Y^c}J^c)X^c)]
\end{aligned}
$$

Making necessary operations and taking $[X^c,Y^c]=\nabla^c_{X^c}Y^c-\nabla^c_{Y^c}X^c$, $(J^c)^2=I$, finally it is shown

$$
\begin{aligned}
4T^{c\widetilde{\nabla}^c}(X^c,Y^c) &= -[J^cX^c,J^cY^c]-[X^c,Y^c]+J^c[X^c,J^cY^c]+J^c[X^c,J^cY^c]\\
&= -N^c_{J^c}(X^c,Y^c).
\end{aligned}
$$

Corollary: Every tangent bundle TM endowed with para-complex structure J^c admits a para-complex torsion-free affine connection.

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